Two-Dimensional Problems in Cartesian Coordinates

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- Uniaxial Tension of a Bar
- Pure Bending of a Beam
- Beam under Uniform Transverse Loading
- River Dam
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- Rectangular Domain with Arbitrary Symmetric Traction Loads

Two-Dimensional Plane Elasticity

- Using the Airy Stress Function approach, it was shown that the plane elasticity formulation with harmonic body force potential reduces to a single governing biharmonic equation.

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = \nabla^4 \psi = \frac{2(1-\kappa)}{1+\kappa}\nabla^2 V$$

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2} + V \ , \ \sigma_y = \frac{\partial^2 \psi}{\partial x^2} + V \ , \ \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

- Boundary conditions need to be satisfied to complete a solution.
- **Inverse or Semi-Inverse Method** is typically applied.

Polynomial Solutions

- In Cartesian coordinates we choose Airy stress function solution of polynomial form
$$\psi(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}x^m y^n$$
- Noted that the three lowest order terms with $m + n \le 1$ do not contribute to the stresses and will therefore be dropped.
- Second order terms will produce a constant stress field
$$\psi(x,y)=A_{20}x^2+A_{11}xy+A_{02}y^2$$
$$\sigma_x=\frac{\partial^2\psi}{\partial y^2}=2A_{02},\ \sigma_y=2A_{20},\ \tau_{xy}=-A_{11}$$
- Third-order terms will give a linear distribution of stress, and so on for higher-order polynomials.
- Terms with $m + n \le 3$ will automatically satisfy the biharmonic equation for any choice of constants A_{mn}.

Polynomial Solutions

- For $m + n \geq 4$, constants A_{mn} will have to be related to have the polynomial satisfy the biharmonic equation. (Specifying additional equations on A_{mn}.)

$$0 = \nabla^4 \psi = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n$$

$$= \sum_{m=4}^{\infty} \sum_{n=0}^{\infty} m(m-1)(m-2)(m-3) A_{mn} x^{m-4} y^n + 2\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} m(m-1)n(n-1) A_{mn} x^{m-2} y^{n-2}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) A_{mn} x^m y^{n-4}$$

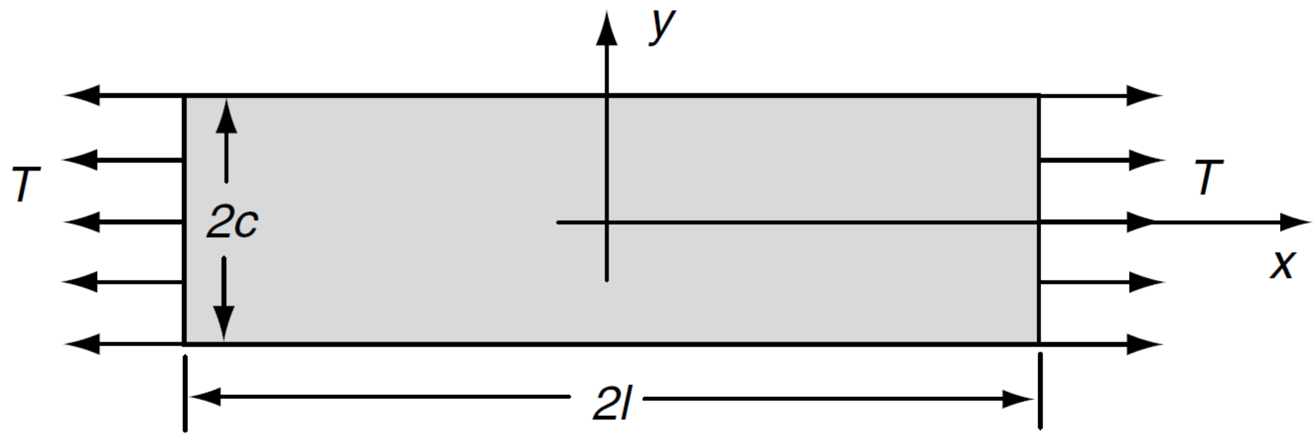
$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left(\begin{array}{l} (m+2)(m+1)m(m-1) A_{m+2,n-2} + 2m(m-1)n(n-1) A_{mn} \\ +(n+2)(n+1)n(n-1) A_{m-2,n+2} \end{array} \right) x^{m-2} y^{n-2}$$

$$\Rightarrow \boxed{\begin{array}{l} (m+2)(m+1)m(m-1) A_{m+2,n-2} + 2m(m-1)n(n-1) A_{mn} \\ +(n+2)(n+1)n(n-1) A_{m-2,n+2} = 0 \end{array}}$$

Polynomial Solutions

- This method produces polynomial stress distributions, and thus would not satisfy general boundary conditions.
- However, we can modify such boundary conditions using Saint-Venant's principle and replace a non-polynomial condition with a statically equivalent loading.
- The solution to the modified problem would then be accurate at points sufficiently far away from the boundary where adjustments were made.
- This formulation is most useful for problems with rectangular domains in which one dimension is much larger than the other. This would include a variety of beam problems.

Example: Uniaxial Tension of a Bar



- Boundary Conditions: $\sigma_x(\pm l, y) = T,\ \tau_{xy}(\pm l, y) = 0; \quad \sigma_y(x, \pm c) = 0,\ \ \tau_{xy}(x, \pm c) = 0$

- Since the boundary conditions specify constant stresses on all boundaries, try a second-order stress function of the form

$$\psi = A_{02} y^2 \ \Rightarrow\ \sigma_x = 2A_{02}\ ,\ \sigma_y = \tau_{xy} = 0$$

- The first boundary condition implies that $A_{02} = T/2$, and all other boundary conditions are identically satisfied. Therefore the **stress field** solution is given by

$$\sigma_x = T\ ,\ \sigma_y = \tau_{xy} = 0.$$

Example: Uniaxial Tension of a Bar

- Displacement Field (Plane Stress)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y) = \dfrac{T}{E} \\ \dfrac{\partial v}{\partial y} = \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x) = -\nu\dfrac{T}{E} \end{cases} \Rightarrow \begin{cases} u = \dfrac{T}{E}x + f(y) \\ v = -\nu\dfrac{T}{E}y + g(x) \end{cases}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy} = \frac{\tau_{xy}}{\mu} = 0 \Rightarrow f'(y) + g'(x) = 0$$

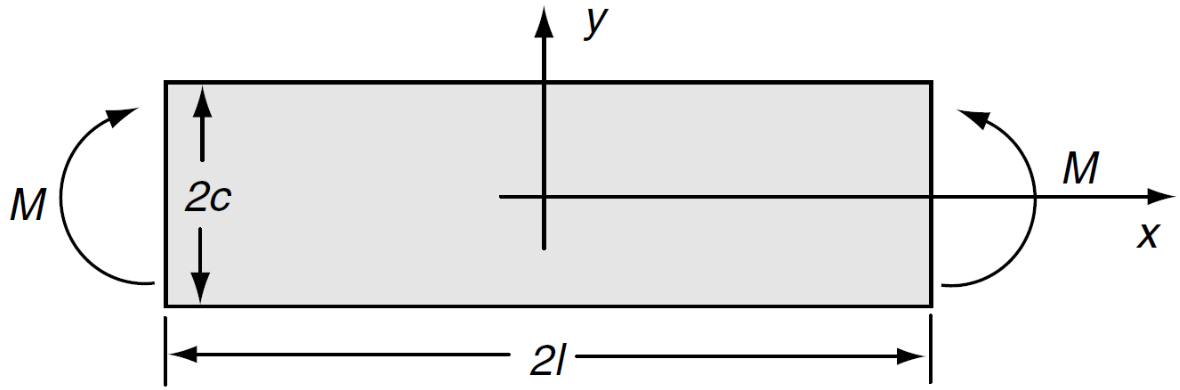
$$\Rightarrow \begin{cases} f(y) = -\omega_o y + u_o \quad \text{. . . } \textit{Rigid-Body Motion} \\ g(x) = \omega_o x + v_o \end{cases}$$

- They do not contribute to the strain or stress fields. Recall that, the displacements are determined from the strain field up to an arbitrary rigid-body motion.

- *"Fixity conditions"* needed to determine these terms explicitly, i.e.

$$u(0,0) = v(0,0) = \omega_z(0,0) = 0 \Rightarrow f(y) = g(x) = 0$$

Example: Pure Bending of a Beam



- Boundary Conditions: $\sigma_y(x,\pm c) = 0$, $\tau_{xy}(x,\pm c) = \tau_{xy}(\pm l, y) = 0$

$$\int_{-c}^{c} \sigma_x(\pm l, y)dy = 0 \; , \; \int_{-c}^{c} \sigma_x(\pm l, y)ydy = -M$$

- Expecting a linear bending stress distribution, try third-order stress function of the form

$$\psi = A_{03}y^3 \;\Rightarrow\; \sigma_x = 6A_{03}y \; , \; \sigma_y = \tau_{xy} = 0$$

- Moment boundary condition implies that $A_{03} = -M/4c^3$, and all other boundary conditions are identically satisfied. Thus the **stress field** is

$$\sigma_x = -\frac{3M}{2c^3}y \; , \; \sigma_y = \tau_{xy} = 0$$

Example: Pure Bending of a Beam

- Displacement Field (Plane Stress)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y) = -\dfrac{3M}{2Ec^3} y \\ \dfrac{\partial v}{\partial y} = \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x) = \nu \dfrac{3M}{2Ec^3} y \end{cases} \Rightarrow \begin{cases} u = -\dfrac{3M}{2Ec^3} xy + f(y) \\ v = \dfrac{3M\nu}{4Ec^3} y^2 + g(x) \end{cases}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \Rightarrow -\frac{3M}{2Ec^3} x + f'(y) + g'(x) = 0$$

$$\Rightarrow \begin{cases} f(y) = -\omega_o y + u_o \\ g(x) = \dfrac{3M}{4Ec^3} x^2 + \omega_o x + v_o \end{cases}$$

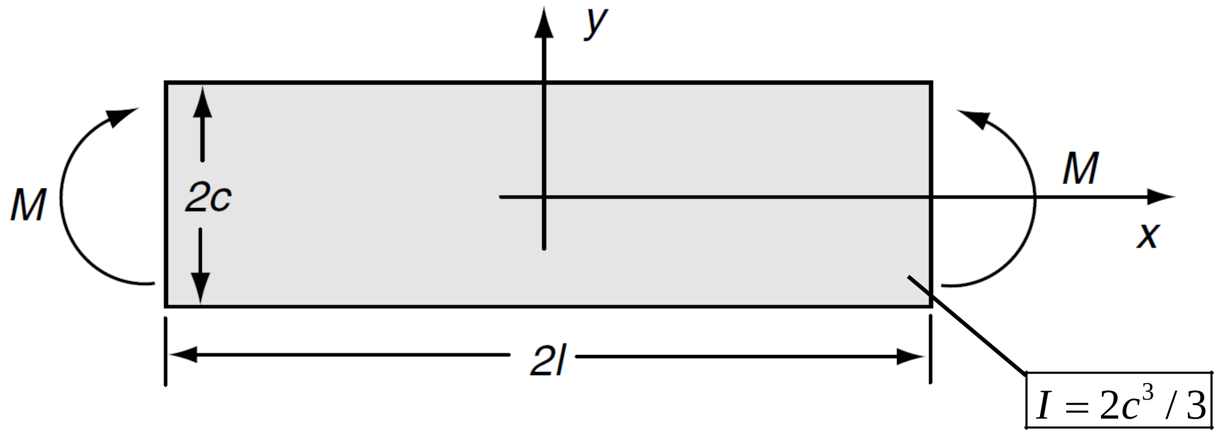
- *"Fixity conditions"* to determine rigid-body motion terms, i.e. a simply supported beam

$$v(\pm l, 0) = 0 \text{ and } u(-l, 0) = 0$$

$$\Rightarrow u_o = \omega_o = 0, \quad v_o = -\frac{3Ml^2}{4Ec^3}$$

$$\Rightarrow \begin{cases} u = -\dfrac{3M}{2Ec^3} xy \\ v = \dfrac{3M}{4Ec^3}\left(\nu y^2 + x^2 - l^2\right) \end{cases}$$

Example: Pure Bending of a Beam



Elasticity Solution

$$\sigma_x = -\frac{M}{I}y,$$

$$\sigma_y = \tau_{xy} = 0;$$

$$u = -\frac{Mxy}{EI},$$

$$v = \frac{M}{2EI}\left(\nu y^2 + x^2 - l^2\right).$$

Strength of Materials Solution

Uses Euler-Bernoulli beam theory to find bending stress and deflection of beam centerline

$$\sigma_x = -\frac{M}{I}y,$$

$$\sigma_y = \tau_{xy} = 0;$$

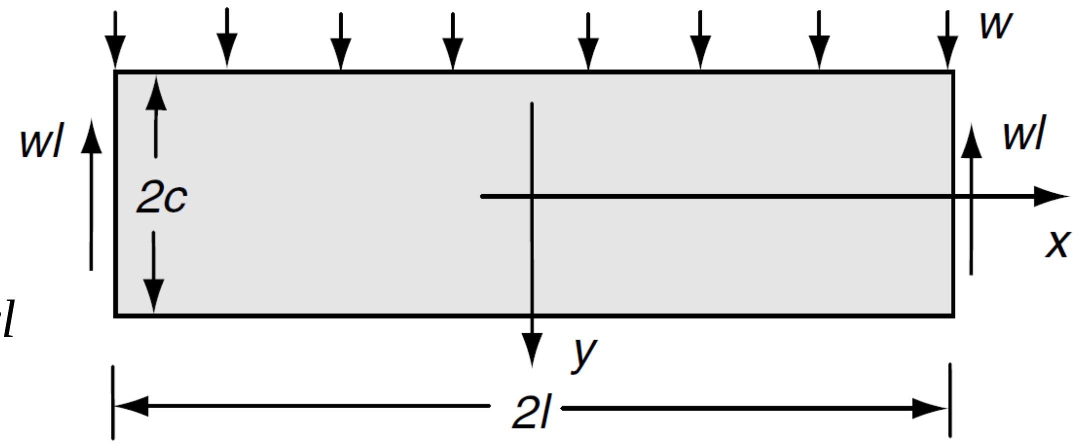
$$v = v(x,0) = \frac{M}{2EI}\left(x^2 - l^2\right).$$

- **Two solutions are identical, with the exception of the *x*-displacements**

Example: Beam under Uniform Transverse Loading

- Boundary conditions

$$\begin{cases} \tau_{xy}(x, \pm c) = 0 \\ \sigma_y(x, c) = 0 \\ \sigma_y(x, -c) = -w \end{cases} \qquad \begin{cases} F_N(\pm l, 0) = 0 \\ M(\pm l, 0) = 0 \\ F_S(\pm l, 0) = \mp wl \end{cases}$$



- Solve by the semi-inverse method
- Analyze the sources of individual stress components and propose an appropriate form for Airy Stress Function

$$M \to \sigma_x, \quad F_S \to \tau_{xy}, \quad w \to \sigma_y, \quad w = \text{constant} \to \sigma_y = f\left(y\right) = \frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow \psi = \frac{x^2}{2} f(y) + x f_1(y) + f_2(y)$$

Example: Beam under Uniform Transverse Loading

- Satisfying the biharmonic equation

$$0=\nabla^4\psi=\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=\frac{x^2}{2}f^{(4)}(y)+xf_1^{(4)}(y)+f_2^{(4)}(y)+2f^{(2)}(y)$$

$$\Rightarrow\begin{cases}f^{(4)}(y)=0\\ f_1^{(4)}(y)=0\\ f_2^{(4)}(y)+2f^{(2)}(y)=0\end{cases}\Rightarrow\begin{cases}f(y)=Ay^3+By^2+Cy+D\\ f_1(y)=Ey^3+Fy^2+Gy\\ f_2^{(4)}(y)=-2f^{(2)}(y)=-2\left(6Ay+2B\right)\end{cases}$$

$$\Rightarrow f_2(y)=-\frac{A}{10}y^5-\frac{B}{6}y^4+Hy^3+Ky^2$$

$$\Rightarrow\psi=\frac{x^2}{2}\left(Ay^3+By^2+Cy+D\right)+x\left(Ey^3+Fy^2+Gy\right)+\left(-\frac{A}{10}y^5-\frac{B}{6}y^4+Hy^3+Ky^2\right)$$

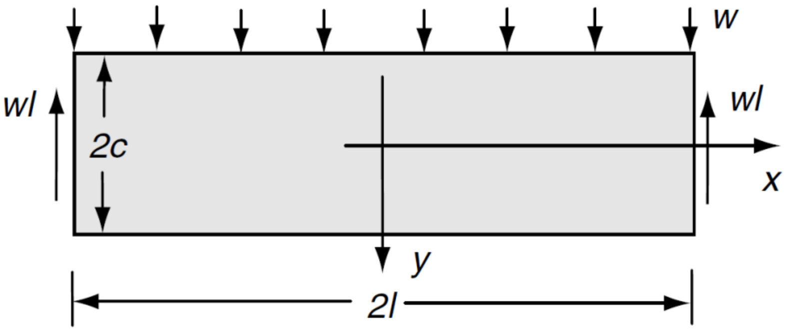
Example: Beam under Uniform Transverse Loading

- Stress field

$$\psi = \frac{x^2}{2}\left(Ay^3 + By^2 + Cy + D\right) + x\left(Ey^3 + Fy^2 + Gy\right) + \left(-\frac{A}{10}y^5 - \frac{B}{6}y^4 + Hy^3 + Ky^2\right)$$

$$\Rightarrow \begin{cases} \sigma_x = \dfrac{\partial^2\psi}{\partial y^2} = \dfrac{x^2}{2}(6Ay + 2B) + x(6Ey + 2F) - 2Ay^3 - 2By^2 + 6Hy + 2K \\ \sigma_y = \dfrac{\partial^2\psi}{\partial x^2} = Ay^3 + By^2 + Cy + D \\ \tau_{xy} = -\dfrac{\partial^2\psi}{\partial x \partial y} = -x(3Ay^2 + 2By + C) - (3Ey^2 + 2Fy + G) \end{cases}$$

- Symmetry property



$$\Rightarrow \begin{cases} \sigma_x(x,y) = \sigma_x(-x,y) \\ \sigma_y(x,y) = \sigma_y(-x,y) \\ \tau_{xy}(x,y) = -\tau_{xy}(-x,y) \end{cases} \Rightarrow E = F = G = 0$$

Example: Beam under Uniform Transverse Loading

- Primary boundary conditions on upper/lower surfaces

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \psi}{\partial y^2} = \dfrac{x^2}{2}(6Ay + 2B) - 2Ay^3 - 2By^2 + 6Hy + 2K \\ \sigma_y = \dfrac{\partial^2 \psi}{\partial x^2} = Ay^3 + By^2 + Cy + D \\ \tau_{xy} = -\dfrac{\partial^2 \psi}{\partial x \partial y} = -x(3Ay^2 + 2By + C) \end{cases}$$

$$\begin{cases} \tau_{xy}(x, \pm c) = 0 \\ \sigma_y(x, c) = 0 \\ \sigma_y(x, -c) = -w \end{cases} \Rightarrow \begin{cases} 3Ac^2 \pm 2Bc + C = 0 \\ Ac^3 + Bc^2 + Cc + D = 0 \\ -Ac^3 + Bc^2 - Cc + D = -w \end{cases} \Rightarrow \begin{cases} A = -\dfrac{w}{4c^3} \\ B = 0 \\ C = \dfrac{3w}{4c} \\ D = -\dfrac{w}{2} \end{cases} \Rightarrow \begin{cases} \sigma_x = -\dfrac{3w}{4c^3}x^2 y + \dfrac{w}{2c^3}y^3 + 6Hy + 2K \\ \sigma_y = -\dfrac{w}{4c^3}y^3 + \dfrac{3w}{4c}y - \dfrac{w}{2} \\ \tau_{xy} = \dfrac{3w}{4c^3}xy^2 - \dfrac{3w}{4c}x \end{cases}$$

Example: Beam under Uniform Transverse Loading

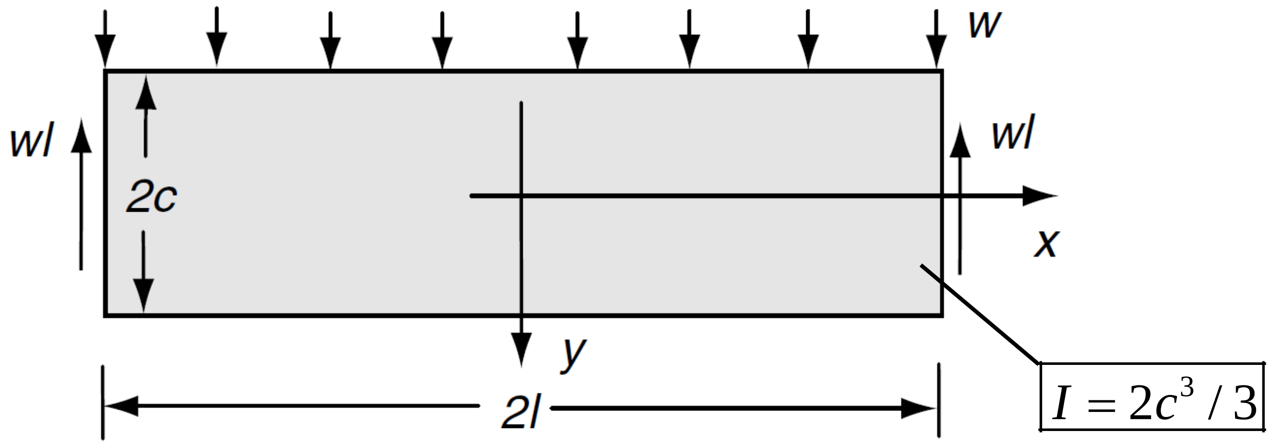
- Minor boundary conditions on ends.
- The end conditions cannot be exactly satisfied. Statically equivalent conditions are sought in terms of Saint-Venant's principle.

$$\begin{cases} \sigma_x = -\dfrac{3w}{4c^3}x^2y + \dfrac{w}{2c^3}y^3 + 6Hy + 2K \\ \sigma_y = -\dfrac{w}{4c^3}y^3 + \dfrac{3w}{4c}y - \dfrac{w}{2} \\ \tau_{xy} = \dfrac{3w}{4c^3}xy^2 - \dfrac{3w}{4c}x \end{cases} \quad \begin{cases} F_N(\pm l,0) = 0 \\ M(\pm l,0) = 0 \\ F_S(\pm l,0) = \mp wl \end{cases} \Rightarrow \begin{cases} \int_{-c}^{c} \sigma_x(\pm l,y)dy = 0 \\ \int_{-c}^{c} \sigma_x(\pm l,y)ydy = 0 \\ \int_{-c}^{c} \tau_{xy}(\pm l,y)dy = \mp wl \end{cases} \Rightarrow \begin{cases} K = 0 \\ H = \dfrac{wl^2}{8c^3} - \dfrac{w}{20c} \\ \text{satisfied.} \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_x = -\dfrac{3w}{4c^3}x^2y + \dfrac{w}{2c^3}y^3 + \left(\dfrac{3wl^2}{4c^3} - \dfrac{3w}{10c}\right)y \\ \sigma_y = -\dfrac{w}{4c^3}y^3 + \dfrac{3w}{4c}y - \dfrac{w}{2} \\ \tau_{xy} = \dfrac{3w}{4c^3}xy^2 - \dfrac{3w}{4c}x \end{cases}$$

$$\boxed{\begin{aligned} \sigma_x &= \frac{3w}{4c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y - \frac{3w}{4c^3}(x^2y - \frac{2}{3}y^3) \\ \sigma_y &= -\frac{w}{2} + \frac{3w}{4c}y - \frac{w}{4c^3}y^3 \\ \tau_{xy} &= -\frac{3w}{4c}x + \frac{3w}{4c^3}xy^2 \end{aligned}}$$

Example: Beam under Uniform Transverse Loading



Elasticity Solution

$$\frac{\sigma_x}{w} = \frac{3y}{4c}\left(\frac{l^2}{c^2} - \frac{x^2}{c^2}\right) + \frac{y^3}{2c^3} - \frac{3y}{10c}$$

$$\frac{\sigma_y}{w} = -\frac{1}{2} + \frac{3y}{4c} - \frac{y^3}{4c^3}$$

$$\frac{\tau_{xy}}{w} = -\frac{3x}{4c} + \frac{3xy^2}{4c^3}$$

Mechanics of Materials Solution

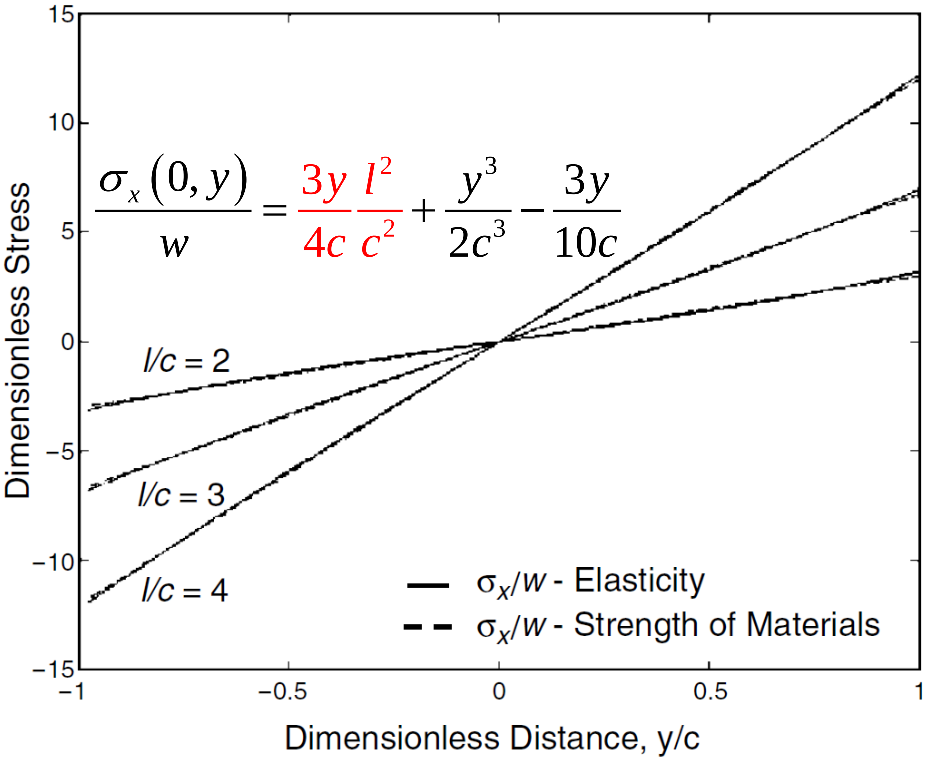
$$\sigma_x = \frac{My}{I} \Rightarrow \frac{\sigma_x}{w} = \frac{3y}{4c}\left(\frac{l^2}{c^2} - \frac{x^2}{c^2}\right)$$

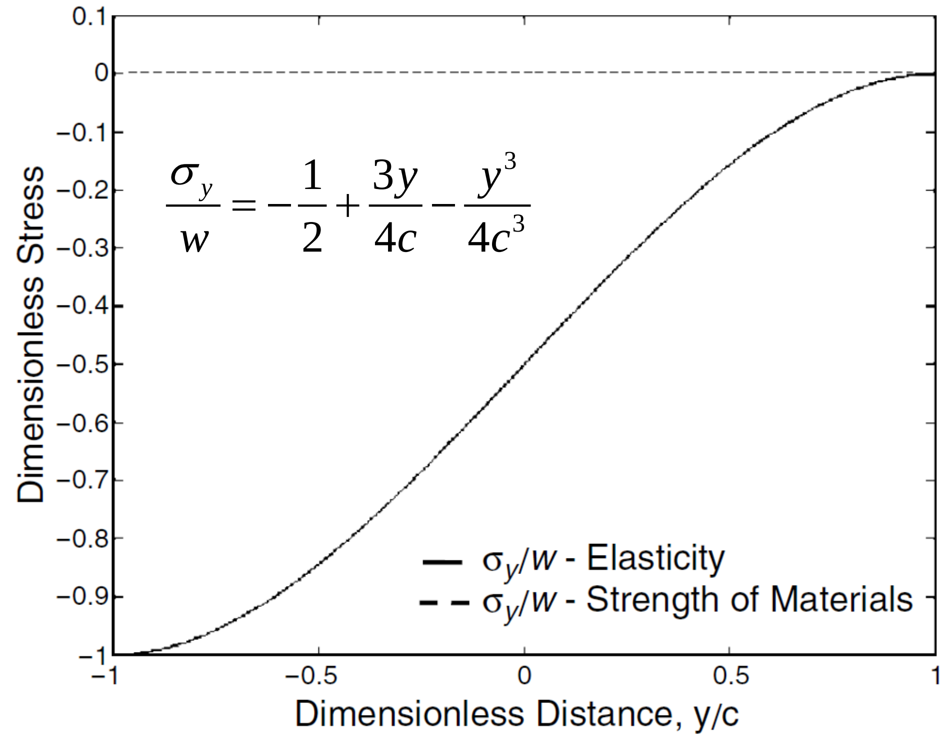
$$\sigma_y = 0$$

$$\tau_{xy} = \frac{VQ}{It} \Rightarrow \tau_{xy} = -\frac{3x}{4c} + \frac{3xy^2}{4c^3}$$

- Shear stresses are identical.
- The relative importance of the correction term in σ_x depends on c/l.
- The transverse normal stress are completely neglected in elementary beam theory.

Example: Beam under Uniform Transverse Loading



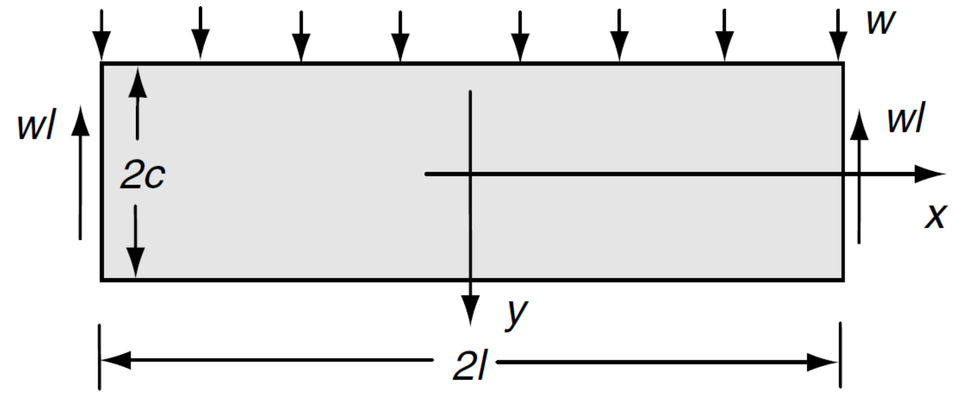


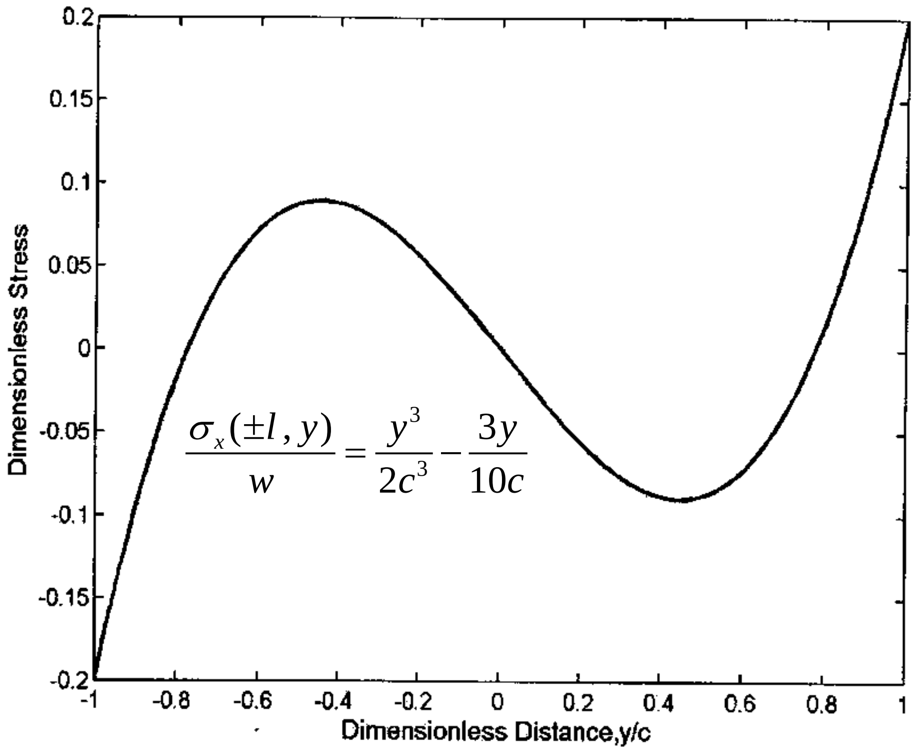
- Maximum differences exist at top and bottom of beam, and actual difference in stress values is *w*/5.
- For *l* >> *c*, the bending stresses will be much greater than *w*, and thus the differences will be relatively small.

- Maximum difference is *w* and this occurs at the top of the beam.
- Again this difference will be negligibly small for *l* >> *c*.
- These results are generally true for beam problems with other transverse loadings.

Example: Beam under Uniform Transverse Loading

- End stress distribution does not vanish and is nonlinear but gives zero resultant force.





Example: Beam under Uniform Transverse Loading

- Displacement Field (Plane Stress)

$$\sigma_x = \frac{w}{2I}\left(l^2 - x^2\right)y + \frac{w}{I}\left(\frac{y^3}{3} - \frac{c^2 y}{5}\right),\ \sigma_y = -\frac{w}{2I}\left(\frac{y^3}{3} - c^2 y + \frac{2}{3}c^3\right),\ \tau_{xy} = -\frac{w}{2I}x\left(c^2 - y^2\right)$$

$$u = \int \varepsilon_x dx = \frac{w}{2EI}\left[\left(l^2 x - \frac{x^3}{3}\right)y + x\left(\frac{2y^3}{3} - \frac{2c^2 y}{5}\right) + \nu x\left(\frac{y^3}{3} - c^2 y + \frac{2c^3}{3}\right)\right] + f(y)$$

$$v = \int \varepsilon_y dy = -\frac{w}{2EI}\left[\left(\frac{y^4}{12} - \frac{c^2 y^2}{2} + \frac{2c^3 y}{3}\right) + \nu\left(l^2 - x^2\right)\frac{y^2}{2} + \nu\left(\frac{y^4}{6} - \frac{c^2 y^2}{5}\right)\right] + g(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\Rightarrow \frac{w}{2EI}\left[\left(l^2 x - \frac{x^3}{3}\right) + x\left(2y^2 - \frac{2c^2}{5}\right) + \nu x\left(y^2 - c^2\right)\right] + f'(y)$$

$$-\frac{w}{2EI}\left[-2x\nu\frac{y^2}{2}\right] + g'(x) = -\frac{(1+\nu)w}{EI}x\left(c^2 - y^2\right)$$

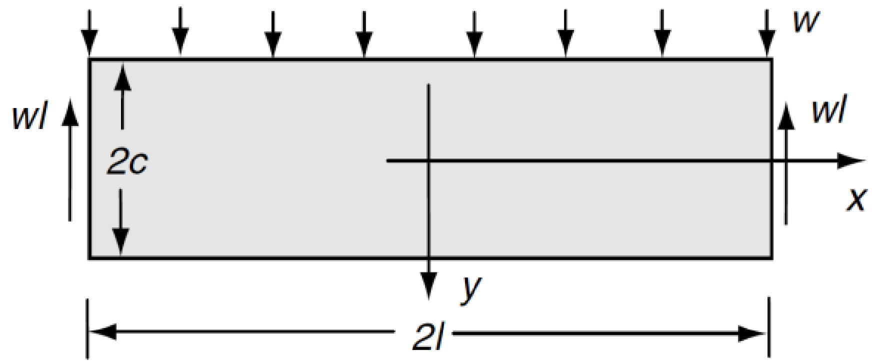
$$\Rightarrow f(y) = \omega_o y + u_o\ ,\quad g(x) = \frac{w}{24EI}x^4 - \frac{w}{4EI}\left[l^2 - \left(\frac{8}{5} + \nu\right)c^2\right]x^2 - \omega_o x + v_o$$

Example: Beam under Uniform Transverse Loading

- Choosing Fixity Conditions

$$u(0,y) = v(\pm l, 0) = 0$$

$$\Rightarrow \begin{cases} u_o = \omega_o = 0, \\ v_o = \dfrac{5wl^4}{24EI}\left[1 + \dfrac{12}{5}\left(\dfrac{4}{5} + \dfrac{\nu}{2}\right)\dfrac{c^2}{l^2}\right] \end{cases}$$



$$\Rightarrow \begin{cases} u = \dfrac{w}{2EI}\left[\left(l^2 x - \dfrac{x^3}{3}\right)y + x\left(\dfrac{2y^3}{3} - \dfrac{2c^2 y}{5}\right) + \nu x\left(\dfrac{y^3}{3} - c^2 y + \dfrac{2c^3}{3}\right)\right] \\ v = -\dfrac{w}{2EI}\left[\left(\dfrac{y^4}{12} - \dfrac{c^2 y^2}{2} + \dfrac{2c^3 y}{3}\right) + \nu\left(l^2 - x^2\right)\dfrac{y^2}{2} + \nu\left(\dfrac{y^4}{6} - \dfrac{c^2 y^2}{5}\right)\right] \\ \quad + \dfrac{w}{24EI}x^4 - \dfrac{w}{4EI}\left[l^2 - \left(\dfrac{8}{5} + \nu\right)c^2\right]x^2 + \dfrac{5wl^4}{24EI}\left[1 + \dfrac{12}{5}\left(\dfrac{4}{5} + \dfrac{\nu}{2}\right)\dfrac{c^2}{l^2}\right] \end{cases}$$

- Good match with elementary mechanics of materials for $l >> c$.

$$v(0,0) = v_{\max} = \frac{5wl^4}{24EI}\left[1 + \frac{12}{5}\left(\frac{4}{5} + \frac{\nu}{2}\right)\frac{c^2}{l^2}\right]$$

Example: River Dam

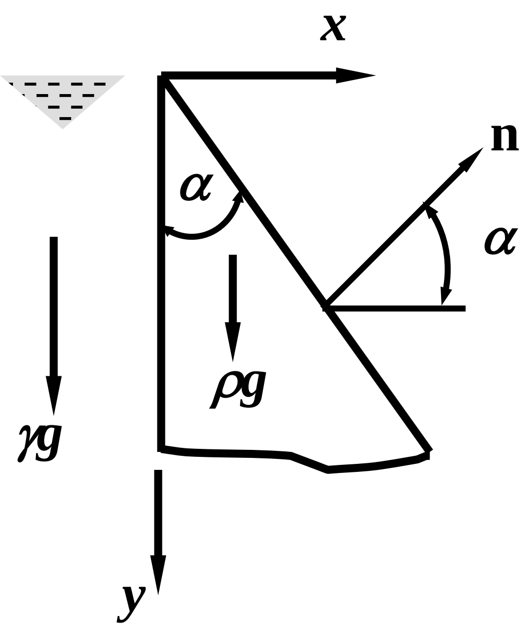
- Stresses occur due to the self-weight of the dam and the water pressure
- Linear elasticity requires

$$\sigma_{\alpha\beta} \propto \rho g,\ \gamma g$$

- **Dimension analysis suggests**

$$\sigma_{\alpha\beta} : \left[\mathrm{Nm}^{-2}\right], \quad \rho g,\ \gamma g : \left[\mathrm{Nm}^{-3}\right]$$

$$\sigma_{\alpha\beta} = A\rho g x + B\rho g y + C\gamma g x + D\gamma g y$$



- Trial Airy stress function

$$\psi = Ax^3 + Bx^2 y + Cxy^2 + Dy^3$$

Example: River Dam

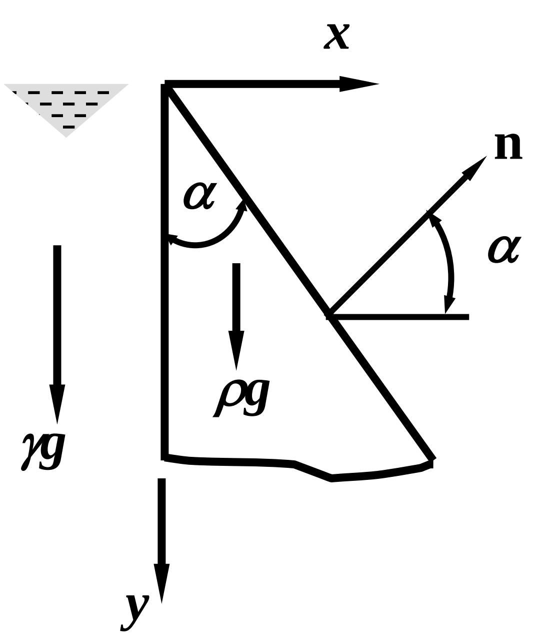
- Stress field

$$\psi = Ax^3 + Bx^2y + Cxy^2 + Dy^3 \Rightarrow \begin{cases} \sigma_x = \dfrac{\partial^2\psi}{\partial y^2} = 2Cx + 6Dy \\ \sigma_y = \dfrac{\partial^2\psi}{\partial x^2} - \rho g y = 6Ax + 2By - \rho g y \\ \tau_{xy} = -\dfrac{\partial^2\psi}{\partial x \partial y} = -2Bx - 2Cy \end{cases}$$

- Boundary conditions

on $x = 0$:

$$\begin{cases} \gamma g y = T_x^{(x=0)} = -(\sigma_x)_{x=0} = -6Dy \\ 0 = T_y^{(x=0)} = (\tau_{xy})_{x=0} = -2Cy \end{cases} \Rightarrow \begin{cases} D = -\dfrac{\gamma g}{6} \\ C = 0 \end{cases}$$



Example: River Dam

- Boundary conditions

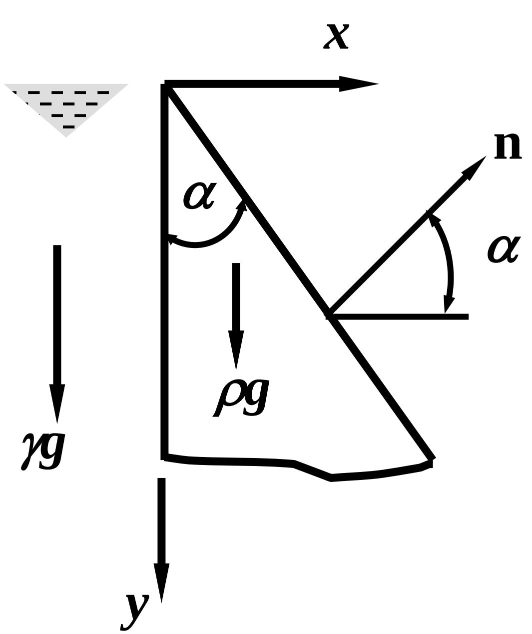
on $x = y\tan\alpha \left(n_x = \cos\alpha, n_y = -\sin\alpha,\right)$:

$$\begin{cases} 0 = T_x^{(\alpha)} = n_x\sigma_x + n_y\tau_{xy} \\ 0 = T_y^{(\alpha)} = n_x\tau_{xy} + n_y\sigma_y \end{cases} \Rightarrow \begin{cases} 0 = \cos\alpha\left(-\gamma gy\right) - \sin\alpha\left(-2By\tan\alpha\right) \\ 0 = \cos\alpha\left(-2By\tan\alpha\right) - \sin\alpha\left(6Ay\tan\alpha + 2By - \rho gy\right) \end{cases}$$

$$\Rightarrow \begin{cases} B = \frac{1}{2}\gamma g\cot^2\alpha \\ A = \frac{1}{6}\rho g\cot\alpha - \frac{1}{3}\gamma g\cot^3\alpha \end{cases}$$

- Final stress field

$$\begin{cases} \sigma_x = -\gamma gy \\ \sigma_y = \left(\rho g\cot\alpha - 2\gamma g\cot^3\alpha\right)x \\ \qquad + \left(\gamma g\cot^2\alpha - \rho g\right)y \\ \tau_{xy} = -\gamma gx\cot^2\alpha \end{cases}$$



Example: River Dam

- Displacement field (Plane Strain)

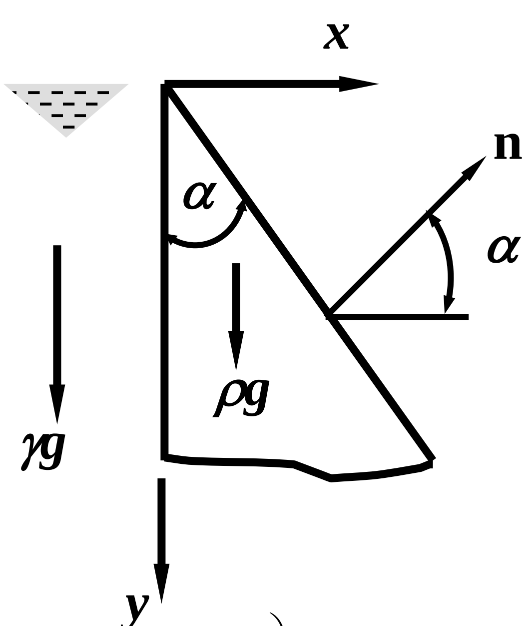
$$\sigma_x = -\gamma g y, \quad \sigma_y = \left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)x + \left(\gamma g \cot^2\alpha - \rho g\right)y, \quad \tau_{xy} = -\gamma g x \cot^2\alpha$$

$$\Rightarrow \begin{cases} \varepsilon_x = -\dfrac{1-\nu^2}{E}\left(\dfrac{\nu}{1-\nu}\left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)x + \gamma g y + \dfrac{\nu}{1-\nu}\left(\gamma g \cot^2\alpha - \rho g\right)y\right), \\ \varepsilon_y = \dfrac{1-\nu^2}{E}\left(\left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)x + \left(\gamma g \cot^2\alpha - \rho g\right)y + \dfrac{\nu}{1-\nu}\gamma g y\right), \quad \varepsilon_{xy} = \dfrac{1+\nu}{E}\sigma_{xy}; \end{cases}$$

$$\Rightarrow \begin{cases} u = \int \varepsilon_x dx = -\dfrac{1-\nu^2}{E}\left(\begin{array}{l} \dfrac{\nu}{2(1-\nu)}\left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)x^2 \\ +\gamma g x y + \dfrac{\nu}{1-\nu}\left(\gamma g \cot^2\alpha - \rho g\right)xy \end{array}\right) + \dfrac{1-\nu^2}{E} f(y) \\ v = \int \varepsilon_y dy = \dfrac{1-\nu^2}{E}\left(\begin{array}{l} \left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)xy \\ +\dfrac{1}{2}\left(\gamma g \cot^2\alpha - \rho g\right)y^2 + \dfrac{\nu}{2(1-\nu)}\gamma g y^2 \end{array}\right) + \dfrac{1-\nu^2}{E} g(x) \end{cases}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \Rightarrow f'(y) + \left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)y$$

$$= -g'(x) + \left(\gamma g - \frac{2-\nu}{1-\nu}\gamma g \cot^2\alpha - \frac{\nu}{1-\nu}\rho g\right)x = \omega_0$$



$$\Rightarrow f(y) = -\frac{1}{2}\left(\rho g \cot\alpha - 2\gamma g \cot^3\alpha\right)y^2 + \omega_0 y + u_0, \quad g(x) = \frac{1}{2}\left(\gamma g - \frac{2-\nu}{1-\nu}\gamma g \cot^2\alpha - \frac{\nu}{1-\nu}\rho g\right)x^2 - \omega_0 x + v_0$$

Fourier Methods

- A more general solution scheme for the biharmonic equation may be found using *Fourier methods*.

$$\psi(x,y) = X(x)Y(y)$$

- Such techniques generally use *separation of variables* along with *Fourier series* or *Fourier integrals*.

$$X = e^{\alpha x},\ Y = e^{\beta y} \Rightarrow \psi(x,y) = e^{\alpha x}e^{\beta y}$$

$$\Rightarrow 0 = \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = \alpha^4 e^{\alpha x}e^{\beta y} + 2\alpha^2\beta^2 e^{\alpha x}e^{\beta y} + \beta^4 e^{\alpha x}e^{\beta y}$$

$$\Rightarrow 0 = \left(\alpha^2 + \beta^2\right)^2 e^{\alpha x}e^{\beta y} \Rightarrow \alpha = \pm i\beta$$

- Polynomial solutions

$$\psi_0 = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 y + C_5 y^2 + C_6 y^3 + C_7 xy + C_8 x^2 y + C_9 xy^2$$

Fourier Methods

- General solutions

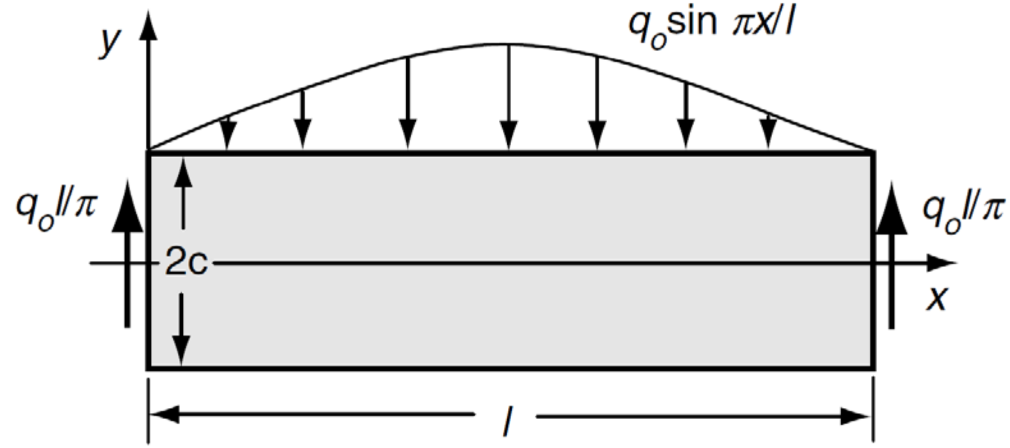
$$\psi = \sin\beta x\left[(A + C\beta y)\sinh\beta y + (B + D\beta y)\cosh\beta y\right]$$
$$+\cos\beta x\left[(A' + C'\beta y)\sinh\beta y + (B' + D'\beta y)\cosh\beta y\right]$$
$$+\sin\alpha y\left[(E + G\alpha x)\sinh\alpha x + (F + H\alpha x)\cosh\alpha x\right]$$
$$+\cos\alpha y\left[(E' + G'\alpha x)\sinh\alpha x + (F' + H'\alpha x)\cosh\alpha x\right]$$
$$+C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 y + C_5 y^2 + C_6 y^3 + C_7 xy + C_8 x^2 y + C_9 xy^2$$

- Using this solution form along with superposition and Fourier series concepts, many problems with complex boundary loadings can be solved.

Example: Beam under Sinusoidal Loading

- Boundary conditions:

$$\begin{cases} \tau_{xy}(x,\pm c)=0 \\ \sigma_y(x,-c)=0 \\ \sigma_y(x,c)=-q_o\sin(\pi x/l) \end{cases} \qquad \begin{cases} \sigma_x(0,y)=\sigma_x(l,y)=0 \\ \int_{-c}^{c}\tau_{xy}(0,y)dy=-q_o l/\pi \\ \int_{-c}^{c}\tau_{xy}(l,y)dy=q_o l/\pi \end{cases}$$



- Trial Airy Stress Function

$$\psi=\sin\beta x\left[(A+C\beta y)\sinh\beta y+(B+D\beta y)\cosh\beta y\right]$$

- Stress field

$$\sigma_x=\beta^2\sin\beta x\begin{bmatrix}(A\sinh\beta y+C(\beta y\sinh\beta y+2\cosh\beta y)\\ +B\cosh\beta y+D(\beta y\cosh\beta y+2\sinh\beta y)\end{bmatrix}$$

$$\sigma_y=-\beta^2\sin\beta x[(A+C\beta y)\sinh\beta y+(B+D\beta y)\cosh\beta y]$$

$$\tau_{xy}=-\beta^2\cos\beta x\begin{bmatrix}(A\cosh\beta y+C(\beta y\cosh\beta y+2\sinh\beta y)\\ +B\sinh\beta y+D(\beta y\sinh\beta y+2\cosh\beta y)\end{bmatrix}$$

Example: Beam under Sinusoidal Loading

- Apply the primary boundary conditions:

$$\begin{cases} \tau_{xy}(x, \pm c) = 0 \\ \sigma_y(x, -c) = 0 \\ \sigma_y(x, c) = -q_o \sin(\pi x / l) \end{cases} \Rightarrow$$

$$\beta = \frac{\pi}{l}, \quad C = \frac{-q_o \sinh \frac{\pi c}{l}}{2\frac{\pi^2}{l^2}\left[\frac{\pi c}{l} + \sinh\frac{\pi c}{l}\cosh\frac{\pi c}{l}\right]}, \quad D = \frac{q_o \cosh \frac{\pi c}{l}}{2\frac{\pi^2}{l^2}\left[\frac{\pi c}{l} - \sinh\frac{\pi c}{l}\cosh\frac{\pi c}{l}\right]},$$

$$A = -D(\beta c \tanh \beta c + 1), \quad B = -C(\beta c \coth \beta c + 1)$$

- The minor boundary conditions are automatically satisfied.
- Bending stress

$$\sigma_x = -\frac{q_o}{2}\sinh\frac{\pi c}{l}\sin\frac{\pi x}{l}\left[\frac{\pi y \cosh\frac{\pi y}{l} + 2l\sinh\frac{\pi y}{l} - \left(\pi c \tanh\frac{\pi c}{l} + l\right)\sinh\frac{\pi y}{l}}{\pi c + l\sinh\frac{\pi c}{l}\cosh\frac{\pi c}{l}}\right.$$

$$\left. + \frac{\pi y \sinh\frac{\pi y}{l} + 2l\cosh\frac{\pi y}{l} - \left(\pi c \coth\frac{\pi c}{l} + l\right)\cosh\frac{\pi y}{l}}{\pi c + l\sinh\frac{\pi c}{l}\cosh\frac{\pi c}{l}}\right].$$

Example: Beam under Sinusoidal Loading

- Bending stress

For the case $l >> c$:

$$D \approx -\frac{3q_o l^5}{4c^3\pi^5}\,,\ C \approx 0\,,\ A \approx -D\,,\ B \approx 0$$

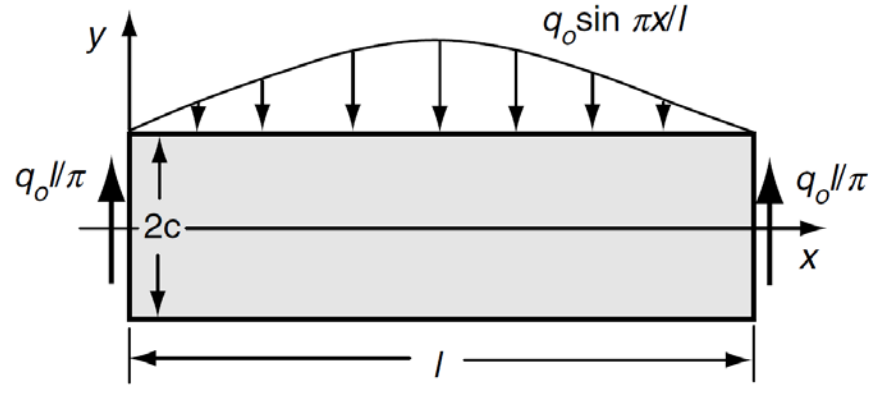
$$\Rightarrow \sigma_x \approx -\frac{3q_o l^3}{4c^3\pi^3}\left(\frac{\pi y}{l}\cosh\frac{\pi y}{l} + \sinh\frac{\pi y}{l}\right)\sin\frac{\pi x}{l}$$

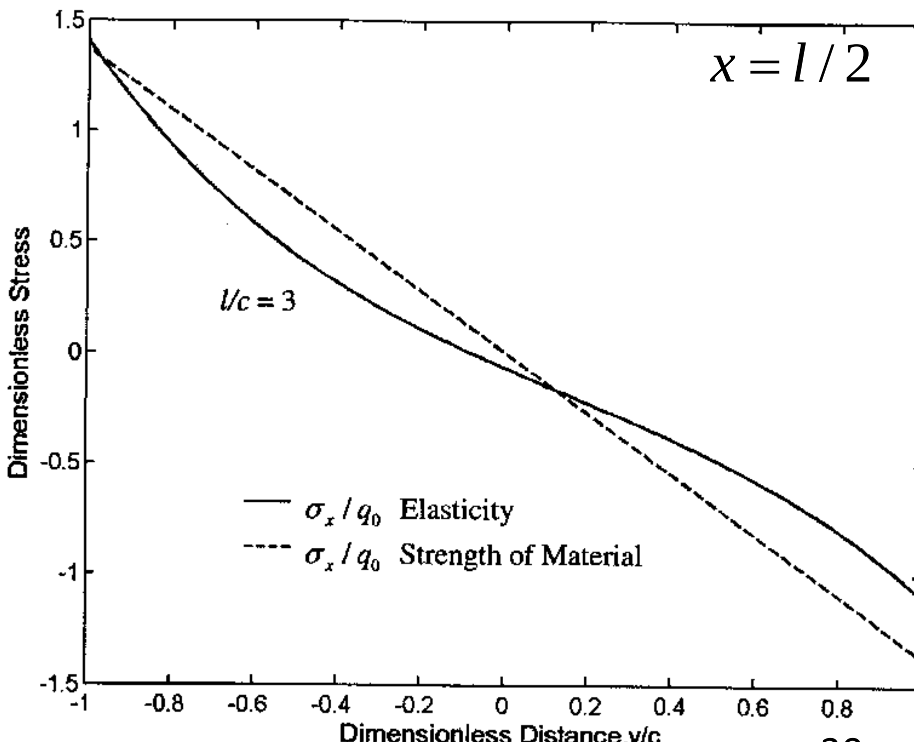
$$\approx -\frac{3q_o l^2}{2c^3\pi^2}\,y\sin\frac{\pi x}{l}$$

- Strength of Materials Theory:

$$\sigma_x = -\frac{My}{I} = -\frac{\frac{q_o l^2}{\pi^2}\sin\frac{\pi x}{l}\ y}{2c^3/3}$$

$$= -\frac{3q_o l^2}{2c^3\pi^2}\,y\sin\frac{\pi x}{l}$$





Example: Beam under Sinusoidal Loading

- Displacement field (Plane Stress)

$$u = -\frac{\beta}{E}\cos\beta x\begin{bmatrix} A(1+\nu)\sinh\beta y + B(1+\nu)\cosh\beta y + C\left((1+\nu)\beta y\sinh\beta y + 2\cosh\beta y\right) \\ +D\left((1+\nu)\beta y\cosh\beta y + 2\sinh\beta y\right) \end{bmatrix} - \omega_o y + u_o$$

$$v = -\frac{\beta}{E}\sin\beta x\begin{bmatrix} A(1+\nu)\cosh\beta y + B(1+\nu)\sinh\beta y + C\left((1+\nu)\beta y\cosh\beta y - (1+\nu)\sinh\beta y\right) \\ +D\left((1+\nu)\beta y\sinh\beta y - (1-\nu)\cosh\beta y\right) \end{bmatrix} + \omega_o y + v_o$$

- Choosing fixity conditions

$$u(0,0) = v(0,0) = v(l,0) = 0 \;\Rightarrow\; \omega_o = v_o = 0\,,\; u_o = \frac{\beta}{E}\left[B(1+\nu) + 2C\right]$$

$$\Rightarrow v(x,0) = \frac{D\beta}{E}\sin\beta x\left[2 + (1+\nu)\beta c\tanh\beta c\right]$$

For the case $l >> c$: $D \approx -\frac{3q_o l^5}{4c^3\pi^5} \;\Rightarrow\; v(x,0) = -\frac{3q_o l^4}{2c^3\pi^4 E}\sin\frac{\pi x}{l}\left[1 + \frac{1+\nu}{2}\frac{\pi c}{l}\tanh\frac{\pi c}{l}\right]$

- Strength of Materials Theory:

$$v(x,0) = -\frac{3q_o l^4}{2c^3\pi^4 E}\sin\frac{\pi x}{l}$$

Rectangular Domain with Arbitrary Symmetric Traction

- Must use series representation for Airy stress function to handle general boundary loading.

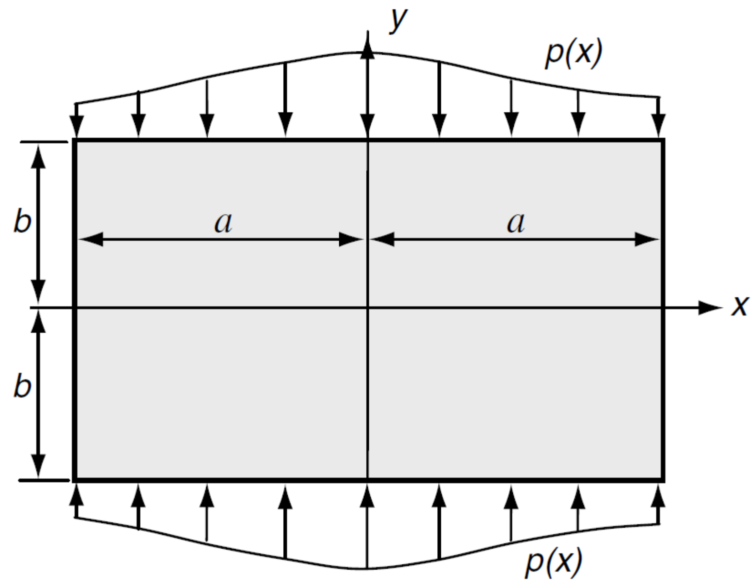
$$\psi = \sum_{n=1}^{\infty} \cos\beta_n x [B_n \cosh\beta_n y + C_n \beta_n y \sinh\beta_n y] + \sum_{m=1}^{\infty} \cos\alpha_m y [F_m \cosh\alpha_m x + G_m \alpha_m x \sinh\alpha_m x] + C_0 x^2$$

$$\Downarrow$$

$$\sigma_x = \sum_{n=1}^{\infty} \beta_n^2 \cos\beta_n x [B_n \cosh\beta_n y + C_n(\beta_n y \sinh\beta_n y + 2\cosh\beta_n y)] - \sum_{m=1}^{\infty} \alpha_m^2 \cos\alpha_m y [F_m \cosh\alpha_m x + G_m \alpha_m x \sinh\alpha_m x]$$

$$\sigma_y = -\sum_{n=1}^{\infty} \beta_n^2 \cos\beta_n x [B_n \cosh\beta_n y + C_n \beta_n y \sinh\beta_n y] + 2C_0 + \sum_{m=1}^{\infty} \alpha_m^2 \cos\alpha_m y [F_m \cosh\alpha_m x + G_m(\alpha_m x \sinh\alpha_m x + 2\cosh\alpha_m x)]$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \beta_n^2 \sin\beta_n x [B_n \sinh\beta_n y + C_n(\beta_n y \cosh\beta_n y + \sinh\beta_n y)] + \sum_{m=1}^{\infty} \alpha_m^2 \sin\alpha_m y [F_m \sinh\alpha_m x + G_m(\alpha_m x \cosh\alpha_m x + \sinh\alpha_m x)]$$



Boundary conditions

$$\begin{cases} \sigma_x(\pm a, y) = 0 \\ \tau_{xy}(\pm a, y) = 0 \end{cases} \quad \begin{cases} \tau_{xy}(x, \pm b) = 0 \\ \sigma_y(x, \pm b) = -p(x) \end{cases}$$

Use Fourier series theory to handle general boundary conditions, and this generates a doubly infinite set of equations to solve for unknown constants in stress function form.

Outline

- Introduction
- Polynomial Solutions
- Uniaxial Tension of a Bar
- Pure Bending of a Beam
- Beam under Uniform Transverse Loading
- River Dam
- Fourier Methods
- Beam under Sinusoidal Loading
- Rectangular Domain with Arbitrary Symmetric Traction Loads